Logarithmic derivative. Local Argument Principle. Rouche Theorem.

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<u>Def</u>. Let $f'(z)$ exists and $f(z) \neq 0$. $\frac{f'}{f}$ is called the <u>logarithmic derivative</u> of f at z.

<u>Heuristics</u>. If $\log f(z)$ is defined, then $\left(\log f(z)\right)' = \frac{f'(z)}{f(z)}$.

<u>Observe</u>: $\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}$. $\frac{\left(\frac{1}{f}\right)'}{1/f} = -\frac{f'}{f}$. $\frac{\left((z-a)^k\right)'}{(z-a)^k} = \frac{k}{z-a}$ $(k \in \mathbb{Z})$

Let γ be a curve, $f \in \mathcal{A}(\gamma)$, $f(z) \neq 0$ $\forall z \in \gamma$.

$\Gamma := f \circ \gamma$ – piecewise differentiable curve.

<u>Observe</u>: ($z(t)$ – parameterization of γ, $f(z(t))$ – of Γ).

$$n(\Gamma, 0) = \frac{1}{2\pi i} \oint_\Gamma \frac{dw}{w} = \frac{1}{2\pi i} \int_a^b \frac{f'(z(t))\, z'(t)}{f(z(t))}\, dt =$$

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\, dz$$ – integral of logarithmic derivative.

Let $\gamma \subset B(z_0, r)$ – closed curve. $\boxed{\mathrm{In}_\gamma := \text{Union of bounded components of } \mathbb{C} \setminus \gamma.}$

<u>Observe</u>: $\mathrm{Clos\,In}_\gamma = \gamma \cup \mathrm{In}_\gamma$, and

$z \notin \mathrm{Clos\,In}_\gamma \Rightarrow z$ is in unbounded component of $\mathbb{C} \setminus \gamma \Rightarrow n(\gamma, z) = 0$.

<u>Local argument principle</u>

<u>Theorem</u>. Let $f \in \mathcal{M}(B(z_0, r))$, γ – closed curve in $B(z_0, r)$. $f(z) \neq 0, \infty$ on γ.

<u>Then</u> $n(f \circ \gamma, 0) = \frac{1}{2\pi i} \int \frac{f'(z)}{f(z)}\, dz = \sum_{z \notin \gamma} \operatorname{ord}(f, z)\, n(\gamma, z)$

<u>Remark</u>. The sum on RHS seems infinite <u>but</u> $z \notin \mathrm{Clos\,In}_\gamma \Rightarrow n(\gamma, z) = 0$, so the sum is finite: compact $\mathrm{Clos\,In}_\gamma$ contains only finitely many zeroes and poles. Also, if z is <u>not</u> zero or pole, $\operatorname{ord}(f, z) = 0$.

<u>Proof</u> Take $r' < r$, so that $\gamma \subset B(z_0, r')$.

Since $\overline{B(z_0, r')} \subset B(z_0, r)$, there are finitely many zeroes and poles of f in $B(z_0, r')$

Let $z_1, z_2, \ldots, z_n$ – zeroes and poles of f in $B(z_0, r')$, with algebraic orders $k_1, \ldots, k_n$ respectively.

Observe that the function $g(z) := (z - z_1)^{-k_1} \cdots (z - z_n)^{-k_n} f(z)$ is

1) $g(z) \in \mathcal{A}(B(z_0, r) \setminus \{z_1, \ldots, z_n\})$, $\lim_{z \to z_i} \frac{f(z)}{(z - z_i)^{k_i}}$ – exists, $\neq 0, \infty$.

1) $g(z) \in A(B(z_0,r) \setminus \{z_1, \dots, z_n\})$, $\lim\limits_{z \to z_j} \frac{f(z)}{(z-z_j)^{k_j}}$ − exists, $\neq 0, \infty$.

2) $\lim\limits_{z \to z_j} g(z) = \dots$

Thus $g(z) \in A(B(z_0,r'))$ $g(z) \neq 0$ $\forall z \in B(z_0, r')$

So $\frac{g'(z)}{g(z)} \in A(B(z_0, r'))$

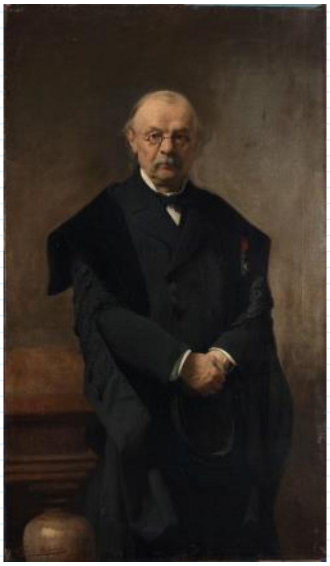
$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{n} \frac{k_j}{(z-z_j)} + \frac{g'(z)}{g(z)} \Rightarrow$$

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} \frac{k_j}{2\pi i} \underbrace{\oint_\gamma \frac{dz}{z-z_j}}_{n(\gamma, z_j)} + \underbrace{\frac{1}{2\pi i}\oint_\gamma \frac{g'(z)}{g(z)}\,dz}_{0 \text{ by Cauchy. } \frac{g'}{g} \in A(B(z_0,r'))}$$

Corollary. Let $f \in B(A \mid B(z_0,r))$, $\gamma \subset B(z_0,r)$ − simple closed curve, N_f − number of zeroes of f inside γ, counting multiplicity. Then $n(f \circ \gamma, 0) = N_f$.

Corollary Let $f \in A(B(z_0,r))$, $\gamma \subset B(z_0,r)$ − closed curve. Then $\forall w \in \mathbb{C}$. $n(f \circ \gamma, w) = \sum h_j\, n(\gamma, z_j(w))$, where $z_j(w)$ are roots of $f(z) = w$ with order h_j. $w \notin f \circ \gamma$.

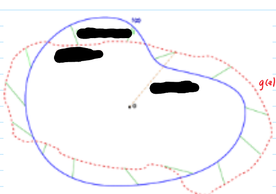
Proof. $n(f \circ \gamma, w) = \frac{1}{2\pi i} \oint_\gamma \frac{f'(\xi)}{f(\xi) - w}\,d\xi$, so we can apply Local Argument Principle to $f(z) - w$ ∎



Eugène Rouché

<u>Theorem</u> (Local Rouché Thm) Let $f, g \in A(B(z_0,r))$, γ - simple closed curve in $B(z_0,r)$, and
$\forall z \in \gamma$ $|f(z) - g(z)| < |f(z)|$.
Then f and g have the same number of zeroes (N_f and N_g) inside of γ, counted with multiplicities.

<u>Heuristics.</u> We ne to prove: $N(f\circ\gamma, 0) = N(g\circ\gamma, 0)$ (Argument principle)
But $g\circ\gamma(t)$ is always at distance $|f(\gamma(t)) - g(\gamma(t))|$ from $f(\gamma(t))$, which is <u>less</u> then distance from $f(\gamma(t))$ to 0.
So it winds around 0 the same number of times!



Proof.

Let $\varphi(z) = \frac{g(z)}{f(z)}$. So $\frac{\varphi'(z)}{\varphi(z)} = \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}$

On γ: $|\varphi(z)-1|<1$.

So $\varphi\circ\gamma \subset \{|\zeta-1|\le 1\}$. $0 \notin \{|\zeta-1|<1\} \Rightarrow n(\varphi\circ\gamma, 0)=0$.

But $0 = n(\varphi\circ\gamma, 0) = \frac{1}{2\pi i}\oint_\gamma \frac{\varphi'(z)}{\varphi(z)}\,dz = \frac{1}{2\pi i}\left(\oint_\gamma \frac{g'(z)}{g(z)}\,dz - \oint_\gamma \frac{f'(z)}{f(z)}\,dz\right) =$

$N_g - N_f$. ∎

Another proof of FTA.

Let $p(z) = a_d z^d + \underbrace{a_{d-1} z^{d-1} + \dots + a_0}_{q(z)}$

Let $f(z) = a_d z^d$.

Then $\lim_{z\to\infty} \frac{q(z)}{f(z)} = \sum_{k=0}^{d-1} \frac{a_k}{a_d} \lim_{z\to\infty} z^{k-d} = 0$.

So for large R, if $|z|=R$ then $\frac{|p(z)-f(z)|}{|f(z)|} = \frac{|q(z)|}{|f(z)|} < 1$.

So, by Rouche applied to $C_R = \{Re^{it}\}$,

$N_p = N_f = d$ ∎